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INVOLVE : (2023)
https://doi.org/10.2140/involve.2023..101

Total difference chromatic numbers of regular infinite graphs

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(Communicated by Kenneth S. Berenhaut)

Given a graph G, a *k-total difference labeling* of the graph is a total labeling f from the set of edges and vertices to the set $\{1, 2, \ldots, k\}$ satisfying $f(\{u, v\}) = |f(u) - f(v)|$ for any edge $\{u, v\}$. If G is a graph, then $\chi_{\text{td}}(G)$ is the minimum k such that there is a k-total difference labeling of G in which no two adjacent labels are identical. We extend prior work on total difference labeling by improving the upper bound on $\chi_{\text{td}}(K_n)$ and also by proving results concerning infinite regular graphs.

By a *k-vertex labeling* of a graph, we mean a function f from the vertices to the positive integers $\{1, 2, \ldots, k\}$ for some k. Similarly, by a *k-edge labeling* of a graph, we mean a function f from the edges to $\{1, 2, \ldots, k\}$ for some k. A *k-total labeling* is a function f from the set of edges and vertices to $\{1, 2, \ldots, k\}$ for some k. A k-vertex labeling is said to be *proper* if no two adjacent vertices share the same label. Similarly, a k-edge labeling is proper if no two edges that share a vertex share a label. A *proper k-total labeling* is a k-total labeling such that its corresponding k-edge labeling is proper, its corresponding k-vertex labeling is proper, and no edge has the same label as either of its vertices.

Ranjan Rohatgi and Yufei Zhang [2020] introduced the idea of a total difference labeling of a graph.

Given a graph G, a *k-total difference labeling* of the graph is a total labeling f from the set of edges and vertices to the set $\{1, 2, \ldots, k\}$ satisfying $f(\{u, v\}) = |f(u) - f(v)|$ for any edge $\{u, v\}$. Recall that a total labeling of a graph is a labeling of both the edges and vertices of a graph. In general, f is a function from the union of the edge set and vertex set of G (denoted by $E(G)$ and $V(G)$, respectively) to the set $\{1, 2, \ldots, k\}$. We will concern ourselves with proper total difference labelings. In a proper k-total difference labeling, f is a function from $V(G) \cup E(G)$ to the set $\{1, 2, \ldots, k\}$ that satisfies the following properties:

MSC2020: primary 05C15; secondary 05C63.
Keywords: graph coloring, chromatic number.

(1) $f(\{u, v\}) = |f(u) - f(v)|$ for any edge $\{u, v\}$.

(2) No two adjacent vertices have the same label. That is, if $\{u, v\}$ is an edge, then $f(u) \neq f(v)$.

(3) No two adjacent edges have the same label. That is, if $\{u, v\}$ and $\{v, w\}$ are edges, then $f(\{u, v\}) \neq f(\{v, w\})$.

(4) No vertex has the same label as an edge incident with it. That is, if $\{u, v\}$ is an edge, then $f(u) \neq f(\{u, v\})$.

Property (1) is the defining property of total difference labelings, while properties (2), (3), and (4) make such a labeling proper. Note that the edge labels of a total difference labeling are determined by the vertex labels. Thus, we will often abuse notation and simply refer to the labeling of the vertices as the total difference labeling. We will typically abbreviate "total difference labeling" to "TDL"; we will also typically omit "proper" when referring to proper TDLs, and unless stated otherwise a TDL may be assumed to be proper.

Rohatgi and Zhang defined $\chi_{\text{td}}(G)$ as the smallest k such that G has a proper k-total difference labeling. They calculated $\chi_{\text{td}}(G)$ for a variety of graphs, including stars, wheels, and helms, as well as providing upper and lower bounds on χ_{td} of other graphs, such as trees and complete graphs. This paper extends their work in three main ways.

First, we provide an essentially complete description of $\chi_{\text{td}}(K_n)$ for complete graphs and use this to bound $\chi_{\text{td}}(G)$ in general for any graph G in terms of its order. To do this, we introduce the idea of a specialized set of numbers we call a *well-spaced row* (occasionally abbreviated to "WSR").

Second, we calculate $\chi_{\text{td}}(G)$ for various well-behaved infinite graphs, including the infinite square lattice. We also provide lower and upper bounds for $\chi_{\text{td}}(G)$ for some other infinite graphs.

Third, we estimate $\chi_{\text{td}}(Q_n)$ for the hypercube graph Q_n.

This paper is divided into five sections. In Section 1, we review aspects of Rohatgi and Zhang's work. We also discuss other similar labeling rules. In Section 2, we introduce the ideas of well-spaced rows and star-elimination, which are major techniques that will be used throughout the rest of the paper. In Section 3, we calculate χ_{td} for various infinite graphs, in particular the square lattice, the hexagonal lattice, the triangular lattice, and the infinite binary tree. We also give upper and lower bounds for the cubic lattice. In Section 4, we introduce the idea of a clone of a graph, and use this to estimate $\chi_{\text{td}}(Q_n)$ of a hypercube.

1. Earlier work

Fundamentally, all of the different labeling schemes under discussion can be thought of as extensions of graph and edge labeling. Classically, given a graph G, the coloring number of G, denoted by $\chi(G)$, is the minimum number of distinct colors needed to label every vertex of G such that no two adjacent vertices are the same color. Instead of using colors, one can use a *labeling function* f from the vertices to $\{1, 2, \ldots, k\}$, which allows a much more natural framework. Thus, when we speak of a "vertex coloring" of a graph we will mean a labeling of a graph's vertices with positive integers. One then defines the coloring number $\chi(G)$ as the minimum k such that there is a labeling function f with the property that $f(u) \neq f(v)$ when $u, v \in V(G)$ are adjacent. In this context, the most notable result is of course the famous four color theorem, which says that any planar graph G satisfies $\chi(G) \leq 4$.

One major result about the behavior of $\chi(G)$ is Brooks's theorem [1941], which states that $\chi(G) \leq \Delta(G) + 1$ for any graph G, with equality if and only if G is a complete graph or a cycle. Here $\Delta(G)$ is the maximum degree of any vertex of G.

Similarly, k-edge labelings have been investigated. Define $\chi'(G)$ to be the minimum k such that G has a proper k-edge labeling. A classic result of [Vizing 1965] states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Let $\chi''(G)$ be the minimum k such that a proper k-total labeling of G exists. One of the major open problems in this area is the *total coloring conjecture*, which states that $\chi''(G) \leq \Delta(G) + 2$.

In the last few years, a variety of papers, such as [Rohatgi and Zhang 2020], have looked at various ways to combine edge colorings and vertex colorings where the edge colors are a function of the vertex colors. Another recent example is [Shiu et al. 2021], which defined an *edge-coloring k-vertex weighting* as a function f from the vertices and edges of G to $\{1, 2, \ldots, k\}$, where, for any edge $\{u, v\}$, $f(\{u, v\}) = f(u) + f(v)$, and the corresponding edge labeling is proper. They then defined $\mu'(G)$ of a graph G as the minimum k such that an edge-coloring k-vertex weighting exists. One can similarly define $\mu''(G)$, which is the minimum k such that there is an edge-coloring k-vertex weighting which is also a proper vertex labeling. It is not hard to show that $\chi_{\text{td}}(G) \geq \mu''(G) \geq \mu'(G)$, since any total difference labeling will also be a total labeling and an edge-coloring k-vertex weighting.

For the remainder of this paper, we will concern ourselves only with total difference labelings and their behavior; recall that we will often omit the word "proper".

One of the basic results of [Rohatgi and Zhang 2020] is a description of total difference labeling just in terms of the vertices by introducing *doubles* and *triples*. Rohatgi and Zhang proved that a total difference labeling is proper if and only if it is a proper k-total labeling and it avoids doubles and triples. What do we mean by doubles and triples?

Let f be a total difference labeling function of G that is not necessarily proper. A double is a pair of adjacent vertices u and v with $f(u) = 2f(v)$ (see Figure 1).

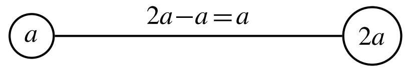

Figure 1. A double.

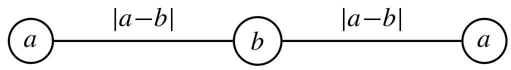

Figure 2. A sandwich.

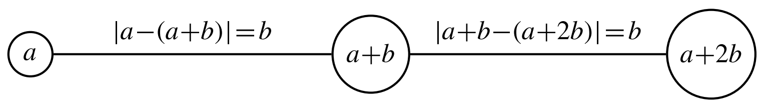

Figure 3. A staircase.

Note that if there exists a double $\{u, v\}$ in G, that is, $f(\{u, v\}) = f(u) - f(v) = 2f(v) - f(v) = f(v)$, then $f(v)$ has the same label as $f(\{u, v\})$, and the labeling due to f is not a total difference labeling.

There are two species of triple. The first is a set of three vertices u, v, and w with u adjacent to v and v adjacent to w satisfying $f(u) = f(w)$. Notice that regardless of what label is given to v, we will then have $f(\{u, v\}) = f(\{v, w\})$. If we have this arrangement, we do not have a proper TDL. We will call this type of triple a *sandwich* (see Figure 2).

The second type of triple is a set of three vertices u, v, and w with u adjacent to v and v adjacent to w such that $f(u)$, $f(v)$, and $f(w)$ form an arithmetic progression. This would also cause $f(\{u, v\}) = f(\{v, w\})$, in which case we would not have a proper TDL. We will call this type of triple a *staircase* (see Figure 3).

Rohatgi and Zhang proved that a total difference labeling is proper if and only if the labeling has no doubles or triples. They did so by using the equivalence of the above description of doubles and triples with respect only to vertex labels and the definition of doubles and triples using both vertex labels and edge labels.

Rohatgi and Zhang proved a variety of bounds on χ_{td} for different graphs. Here, we summarize the bounds we use or improve on this paper.

Proposition 1 [Rohatgi and Zhang 2020, Proposition 2.11]. *Let G' be a subgraph of G. Then $\chi_{\text{td}}(G') \leq \chi_{\text{td}}(G)$.*

Proposition 2 [Rohatgi and Zhang 2020, Proposition 2.5]. *Let G be a graph with n vertices. Then $\chi_{\text{td}}(G) \leq 3^{n-1}$.*

Due to Proposition 1, Proposition 2 is an upper bound on χ_{td} of the complete graph K_n, which is the graph with the highest χ_{td} of the graphs with n vertices (as all such graphs are subgraphs of K_n). We will in the next section construct a substantially better bound on $\chi_{\text{td}}(K_n)$.

Proposition 3 [Rohatgi and Zhang 2020, Theorem 3.1]. *Let $n \geq 4$ and let P_n be the path on n vertices. Then $\chi_{\text{td}}(P_n) = 4$.*

Proposition 4 [Rohatgi and Zhang 2020, Theorem 3.2]. *Let $n > 2$ and let C_n be the cycle on n. Then $\chi_{\text{td}}(C_n) = 4$ if $n \equiv 0 \pmod 3$ and $\chi_{\text{td}}(C_n) = 5$ otherwise.*

Proposition 5 [Rohatgi and Zhang 2020, Theorem 4.1]. *Let $K_{1,m}$ be the star graph with m neighbors of the central vertex. Then $\chi_{\text{td}}(K_{1,m}) = m+1$ when m is even and $\chi_{\text{td}}(K_{1,m}) = m+2$ when m is odd.*

Although not mentioned in [Rohatgi and Zhang 2020], it is worth noting that it is an immediate consequence of Propositions 4 and 5 together, with Brooks's theorem, that, for any connected graph G with more than one vertex, one has $\chi(G) < \chi_{\text{td}}(G)$.

They also proved a lower bound for a large class of graphs.

Proposition 6 [Rohatgi and Zhang 2020, Proposition 2.8]. *Let G be a graph with n vertices where the diameter of G is at most 2. Then $\chi_{\text{td}}(G) \geq n$, and all the vertex labels of G must be distinct in any total difference labeling of G.*

2. Well-spaced rows and star-elimination

We define a *well-spaced row* to be a set of positive integers such that no element is twice another element and no three elements form an arithmetic progression. Well-spaced rows are useful in finding upper bounds on $\chi_{\text{td}}(G)$, as their elements avoid doubles and staircases.

A *minimal well-spaced row* is a well-spaced row that, given a finite cardinality, has the least possible maximum element. This is different from a *greedy well-spaced row*, which is generated from the *greedy well-spaced row algorithm*. The greedy well-spaced row algorithm takes in a number of elements n and outputs a well-spaced row in the following manner. At each step, the algorithm appends the least positive integer such that the resulting set is a well-spaced row; it repeats until the set, a well-spaced row, has n elements.

Closely connected to the idea of a well-spaced row is that of a *nonaveraging set*. A nonaveraging set is a set S of nonnegative integers that includes 0 such that no element of S is the average of two other elements of S. Nonaveraging sets are equivalent to well-spaced rows with the addition of the element 0. In other words, if a set S is a nonaveraging set, then $S - \{0\}$ is a well-spaced row. If R is a well-spaced row, then $R \cup \{0\}$ is a nonaveraging set. This is because an illegal double in a well-spaced row is equivalent to an illegal arithmetic progression with first element 0 in a nonaveraging set. Nonaveraging sets have been previously studied. See, for example, [Moser 1953]. The greedy approach to producing nonaveraging sets was also studied in [Odlyzko and Stanley 1978].

Note that the greedy algorithm does not in general produce a minimal well-spaced row. For example, the greedy well-spaced row of length 4 is $\{1,3,4,9\}$, but $\{1,3,7,8\}$ is also a well-spaced row of length 4, with greatest element 8, which is less than 9.

An immediate consequence of the greedy well-spaced row construction is the following.

Theorem 7. *For any graph G with at most n vertices,* $\chi_{\text{td}}(G) \le 3^{\lceil \log_2 n \rceil}$.

Proof. Assume G has at most n vertices. Let k be the smallest positive integer such that $2^k \ge n$. Then we may label the vertices of G with a subset of the labels of the greedy well-spaced row with 2^k elements, whose largest element is precisely 3^k. □

Notice that

$$3^{\lceil \log_2 n \rceil} < 3^{1+\log_2 n} = 3n^{\log_2 3},$$

and so Theorem 7 gives a much tighter bound than Proposition 2, which gave a bound exponential in n.

Star-elimination is a method that can be used to find lower bounds on the total difference labeling numbers of regular infinite graphs. Recall that each vertex in a regular graph has the same degree. In an infinite regular graph G, each vertex is the center of a star subgraph $K_{1,\Delta}$ of G, where Δ is the degree of each vertex in the graph. Because $K_{1,\Delta}$ is a subgraph of an infinite Δ-regular graph Ω, Proposition 1 yields the quick lower bound $\chi_{\text{td}}(\Omega) \ge \chi_{\text{td}}(K_{1,\Delta})$, which is either $\Delta+1$ or $\Delta+2$, as per Proposition 5. This lower bound is a starting point for the main part of the star-elimination method.

Given a Δ-regular infinite graph, assume $x = \Delta = \chi_{\text{td}}(G)$. We then try to show that it is impossible to label the relevant star subgraph $S = K_{1,\Delta}$ given this value and given the restriction that all labels must be distinct, as per Proposition 6. If a contradiction arises, we set $x+1 = \chi_{\text{td}}(G)$ and repeat until this procedure fails, at which point the star-elimination method has produced a lower bound on the true value of $\chi_{\text{td}}(G)$. In most of our cases, the star-elimination method will produce the best-possible lower bound.

Let v be a vertex in G. Since G is Δ-regular, there must be exactly Δ vertices connected to v. This vertex will have a label, l, in the set of $\{1,2,3,\dots,x\}$.

We will now assign a label to each vertex adjacent to v. We cannot use the label l for any of them, nor can we use $2l$ or $l/2$ since the vertex and v would form a double, resulting in an improper labeling. We also cannot use both $l-a$ and $l+a$ for two vertices and any a, since $l-a$, l, and $l+a$ would form a staircase. Finally, we can't use the same label twice in any of the adjacent vertices since they would form a sandwich around v. Write down all the positive integers less than l in a list, which we will call L_1, in descending order. Write down all the integers greater than l and less than or equal to x in a similar list, called L_2. If L_{ji} is index i of list L_j, we know we can use either L_{1i} or L_{2i}, but not both. In addition, we can't use L_{ji} if it is either $l/2$, l, $2l$, or if it is less than 1 or greater than x. For each value of i, if either L_{1i} or L_{2i} exists, we can arbitrarily label one of the vertices adjacent to v with one of them. This is the maximum number of vertices we can label for this l. If we are not able to label all of the adjacent vertices with this method for any value of l, then $\chi_{\text{td}}(G) > x$, since you must need more labels to complete a proper labeling. If we are able, x is a definite lower bound for $\chi_{\text{td}}(G)$, since all smaller x cannot be $\chi_{\text{td}}(G)$.

To find a contradiction in $x = \chi_{\text{td}}(G)$, we assume each value k from 1 to x is the label of the center of a star S, because in a regular infinite graph each label is assigned to the center of some such S.

In each case, we find the number of vertex labels that could neighbor the center of S. We start with a given k which is at most x, and assume that k is the center of a star. We then list all the possible labels, from 1 to x. We first note that k itself is removed immediately from this list, as the center's label is k, so none of its neighbors can be labeled k.

When k is even we may also remove $k/2$ from our list, as it would create a double with k. Similarly, we may remove $2k$. Note that we may have only one element from each pair of numbers that produces a triple with k as the second element. For example, if we have assumed 4 to be the center label, we must eliminate one of each of 1 and 7, 2 and 6, and 3 and 5 — note that 2 would have already been removed in the previous step. If the number of available labels remaining is less than the number of vertices Δ neighboring the center of S, then k cannot be a vertex label. We repeat this procedure for all labels from 1 to x, marking each label as either "possible" or "impossible". The order in which we check the labels from 1 to x is tactical, as eliminating some values of k will depend on the prior elimination of other values of k. If at any point the total number of "possible" labels for vertices is less than $\Delta+1$, then we have produced a contradiction and may conclude that $\chi_{\text{td}}(G) > x$.

A more formal way of expressing star-elimination is as follows. Given a finite set of positive integers A, and j an element of A, we will say that a subset B of A is j-acceptable if B does not contain $2j$, $j/2$, or a three-term arithmetic sequence with j as the middle term. We say that j is k-star-vulnerable from A if the cardinality of any j-acceptable subset of A is strictly less than k.

A star-elimination sequence with respect to k and n then is a sequence of sets, $S_0, S_1, S_2, \ldots, S_m$, satisfying three properties:

(1) $S_0 = \{1, 2, 3, \ldots, n\}$.

(2) $S_i \subset S_{i-1}$ for any $0 < i \leq m$.

(3) If $j \in S_{i-1}/S_i$ then j is k-star vulnerable.

The star-elimination method is essentially the observation that if there is a star-elimination sequence with respect to k and n, then there cannot be a k-regular total difference labeling with all labels less than or equal to n.

3. Total difference labeling of infinite graphs

We begin by finding the total difference chromatic number of the infinite square lattice, which we denote by Ω_S.

Theorem 8. *We have* $\chi_{\text{td}}(\Omega_S) = 8$.

Proof. First, we use star-elimination to prove that $\chi_{\text{td}}(\Omega_S) \geq 8$. Then we use well-spaced rows to show that $\chi_{\text{td}}(\Omega_S) \leq 8$.

Let us first find a lower bound using star-elimination. Because, by Proposition 5, $\chi_{\text{td}}(K_{1,4}) = 4+1 = 5$ (note that each vertex is the center of a $K_{1,4}$), we have a lower bound of 5, so $\chi_{\text{td}}(\Omega_S) \geq 5$. Assume $\chi_{\text{td}}(\Omega_S) = 5$. Now using star-elimination it is straightforward to show that $\chi_{\text{td}}(\Omega_S) \neq 5$ and $\chi_{\text{td}}(\Omega_S) \neq 6$, so $\chi_{\text{td}}(\Omega_S) \geq 7$.

We now demonstrate star-elimination to show that $\chi_{\text{td}}(\Omega_S) > 7$. Begin by assuming $\chi_{\text{td}}(\Omega_S) = 7$. In this case, we will need to eliminate three of the seven potential labels $\{1, 2, 3, 4, 5, 6, 7\}$ to show a contradiction.

(1) Assume that 4 is included somewhere in a 7-TDL of Ω_S. The vertex v with label 4 has four neighbors, each necessarily distinct (to avoid sandwiches). We first remove 4 from the list of possible labels, as v cannot be adjacent to a vertex labeled 4. We also remove $\frac{4}{2} = 2$, leaving $\{1, 3, 5, 6, 7\}$. We finally remove one of 1 and 7 as well as one of 3 and 5. This leaves only three possible labels for the four neighbors of v, which is a contradiction, so no vertex can be labeled with 4.

(2) Assume that 3 is the label of some vertex v in Ω_S. We remove 3, 4, and $3 \times 2 = 6$ from the list of possible labels of the neighbors of v, leaving $\{1, 2, 5, 7\}$; we then remove one of 1 and 5, again leaving only three labels for the four neighbors of v, which is a contradiction, so no vertex can be labeled with 3.

(3) Assume that 6 is the label of some vertex v in Ω_S. We remove 3, 4, and 6 from the list of possible labels of the neighbors of v, leaving $\{1, 2, 5, 7\}$. We also remove one of 5 and 7, leaving three labels for the neighbors of v, which is a contradiction, so no vertex can be labeled with 6.

Because this leaves only four distinct labels for a TDL of Ω_S, we have a contradiction and $\chi_{\text{td}}(\Omega_S) > 7$. Thus, $\chi_{\text{td}}(\Omega_S) \geq 8$.

We now find an upper bound using well-spaced rows. Clearly each star subgraph $K_{1,4}$ of Ω_S uses five distinct labels. The greatest element of a minimal well-spaced row with five elements is 10, as in $W = \{1, 3, 4, 9, 10\}$ (which incidentally is the same as the greedy well-spaced row of five elements). Labeling a "row" of vertices in Ω_S (hence the name "well-spaced rows") with the elements of W, then

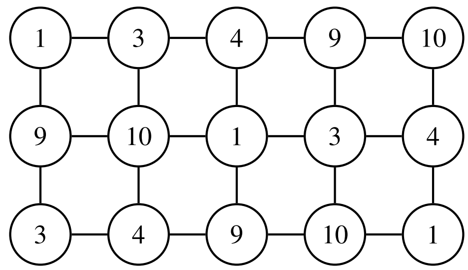

Figure 4. Subgraph of Ω_S and corresponding 10-total difference labeling.

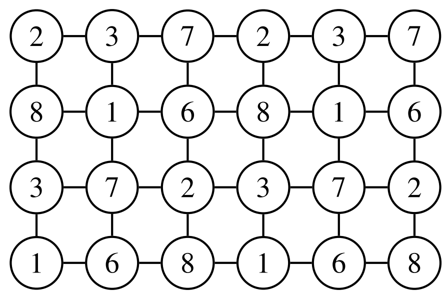

Figure 5. Subgraph of Ω_S and corresponding 8-total difference labeling.

labeling an adjacent row in the same way but with labels shifted two vertices in either direction, and so on, gives a 10-TDL of Ω_S, showing $\chi_{\mathrm{td}}(\Omega_S) \leq 10$ (see Figure 4).

We can improve our weak upper bound from 10 by increasing the number of distinct labels being used. The six labels $\{1,2,3,6,7,8\}$ contain two potential doubles and two potential staircases. However, they can be arranged to form a valid 8-total difference labeling of Ω_S, as shown in Figure 5. □

Finding the total difference chromatic numbers of the infinite hexagonal and triangular lattices is similar. We denote the infinite hexagonal lattice by Ω_H and follow a procedure similar to the one used for Ω_S, using well-spaced rows and star-elimination for lower and upper bounds.

Theorem 9. *We have* $\chi_{\mathrm{td}}(\Omega_H) = 7$.

Proof. We first find a lower bound on $\chi_{\mathrm{td}}(\Omega_H)$ using star-elimination. Because Ω_H is 3-regular, each vertex is the center of a $K_{1,3}$; we therefore have a lower bound of $\chi_{\mathrm{td}}(K_{1,3}) = 3+2 = 5$. So we first assume $\chi_{\mathrm{td}}(\Omega_H) = 5$. It is again straightforward to show a contradiction in this case, so we will start by contradicting that $\chi_{\mathrm{td}}(\Omega_H) = 6$.

(1) Assume 3 appears in a 6-TDL of Ω_H. We eliminate 3 and 6, as well as one of 1 and 5 and one of 2 and 4, leaving two possible labels for its neighbor, which is a contradiction; 3 does not appear.

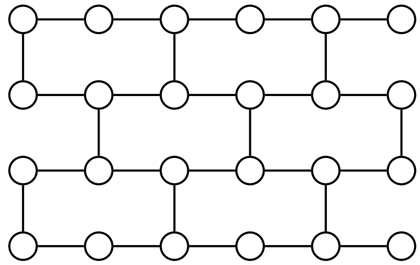

Figure 6. Representation of Ω_H as a subgraph of Ω_S.

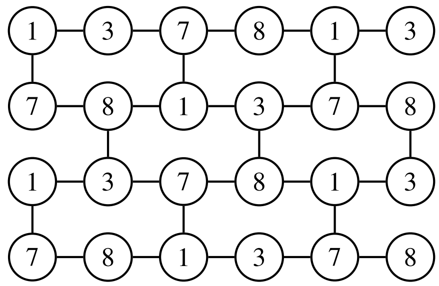

Figure 7. Upper-bound 8-TDL of subgraph Ω_H.

(2) Assume 2 appears. We remove 1, 2, 3, and 4, leaving only 5 and 6, which is a contradiction; 2 does not appear.

(3) Assume 5 appears. We remove 2, 3, and 5, as well as one of 4 and 6, leaving two possible neighboring labels, which is a contradiction; 5 does not appear.

We are left with the three labels $\{1,4,6\}$ for a 6-TDL of Ω_H, which is impossible. Therefore $\chi_{\text{td}}(\Omega_H) > 6$. Star-elimination fails to produce a contradiction for $\chi_{\text{td}}(\Omega_H) = 7$, so $\chi_{\text{td}}(\Omega_H) \geq 7$.

Recall that each vertex of Ω_H is the center of a $K_{1,3}$. The greatest element of a minimal well-spaced row with four elements is 8, as in $W = \{1,3,7,8\}$. The construction of "rows" is not as obvious for Ω_H as it is for Ω_S, but we can structure Ω_H as a subgraph of Ω_S as follows: Construct Ω_S, then remove alternating vertical edges within a row, then shift horizontally by one vertex, repeat in the adjacent rows, and so on, as shown below in Figure 6. Assign a minimal well-spaced row with four elements, such as W, to each row of the graph, shifted horizontally by two vertices in each adjacent row, as in the upper-bound construction for $\chi_{\text{td}}(\Omega_S)$ in Figure 4; see Figure 7 for a subgraph of Ω_H with this labeling. Notice that this results in pairs of labels in each "column" of vertices, which is a useful pattern.

We now know $7 \leq \chi_{\text{td}}(\Omega_H) \leq 8$, and in fact we can find a construction with $\chi_{\text{td}}(\Omega_H) = 7$, using all seven labels $\{1,2,3,4,5,6,7\}$. We do this by labeling the

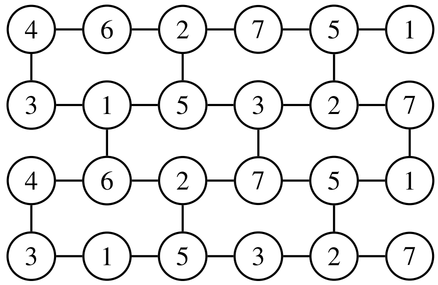

Figure 8. Subgraph of Ω_H with 7-TDL. The pattern is apparent in columns of vertices.

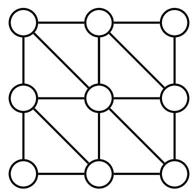

Figure 9. Representation of Ω_T as a graph containing Ω_S as a subgraph.

columns of vertices with the pairs of labels $\{4,3\}$, $\{6,1\}$, $\{2,5\}$, $\{7,3\}$, $\{5,2\}$, and $\{1,7\}$, as shown in Figure 8, similar to the construction in Figure 7. □

We now find $\chi_{\text{td}}(\Omega_T)$, where Ω_T is the infinite triangular lattice.

Theorem 10. *We have* $\chi_{\text{td}}(\Omega_T) = 12$.

Proof. We will again find lower and upper bounds using well-spaced rows and star-elimination. It can be visually useful to restructure Ω_T such that Ω_S is clearly a subgraph of Ω_T, which it is (analogously to the manner in which we represented Ω_H as a subgraph of Ω_S). A simple way of doing this is to construct Ω_S and connect the top-left and bottom-right vertices of each C_4 by an edge, as shown in Figure 9.

Since each vertex of Ω_T is the center of a $K_{1,6}$ (Ω_T is 6-regular) and $\chi_{\text{td}}(K_{1,6}) = 7$, $\chi_{\text{td}}(\Omega_T) \geq 7$, we will need to use seven distinct labels. In fact, we can do better: since Ω_S is a subgraph of Ω_T, we have $\chi_{\text{td}}(\Omega_T) \geq \chi_{\text{td}}(\Omega_S) = 8$.

Star-elimination quickly increases the lower bound on $\chi_{\text{td}}(\Omega_T)$ to 11. We will now use star-elimination to increase the lower bound to 12.

Assume $\chi_{\text{td}}(\Omega_T) = 11$. We will need to show that there are only six possible labels for the seven vertices in each $K_{1,6}$ subgraph of Ω_T. We do this by eliminating six of the eleven possible labels for the neighbors of the central vertex of an arbitrary $K_{1,6}$ in a hypothetical 11-TDL of Ω_T.

(1) Assume 5 appears. Then we eliminate 5 and 10; we also eliminate one of 1 and 9, 2 and 8, 3 and 7, and 4 and 6, leaving 11 as well as one from each of these

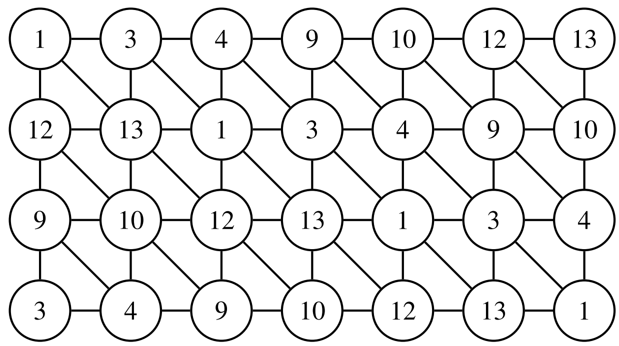

Figure 10. Subgraph of Ω_T with 13-TDL. Each row uses the elements of the minimal 7-element WSR.

pairs, leaving only five possible labels for the six neighbors of 5; therefore, 5 does not appear.

(2) Assume 6 appears. Eliminate 3, 5, and 6. Also eliminate one of 1 and 11, 2 and 10, and 4 and 8. This leaves 7 and 9, plus one of each of the three pairs just checked, leaving only five possible neighbor labels; so 6 does not appear.

(3) Assume 4 appears. Eliminate 2, 3, 4, 5, 6, and 8. Also eliminate one of 1 and 7. This leaves 9, 10, and 11, plus one of 1 and 7; so 4 does not appear.

(4) Assume 9 appears. Eliminate 4, 5, and 6, as well as one of 7 and 11 and 8 and 10. This leaves 1, 2, and 3, plus two of 7, 8, 10, and 11; so 9 does not appear.

(5) Assume 2 appears. Eliminate 1, 2, 4, 5, 6, and 9. This leaves 3, 7, 8, 10, and 11; so 2 does not appear.

(6) Assume 7 appears. Eliminate 2, 4, 5, 6, and 9. Also eliminate one of 3 and 11. This leaves 1, 8, 9, 10, and one of 3 and 11; so 7 does not appear.

We have now eliminated six labels, so $\chi_{\text{td}}(\Omega_T) > 11$.

We use minimal well-spaced rows to find an upper bound on $\chi_{\text{td}}(\Omega_T)$. Because a TDL of Ω_T requires seven distinct labels, we will use the (unique) minimal well-spaced row with seven labels $\{1, 3, 4, 9, 10, 12, 13\}$. We can apply this WSR to Ω_T as in Figure 10.

We can construct a 12-TDL of Ω_T using the labels $\{1, 2, 3, 4, 5, 7, 9, 10, 11, 12\}$; see Figure 11. □

We also have a result for the graph obtained from the cubic lattice. Set Ω_{Q_3} to be the cubic lattice graph.

Theorem 11. *We have* $12 \le \chi_{\text{td}}(\Omega_{Q_3}) \le 13$.

Here the upper bound is obtained from a well-spaced row, and the lower bound is obtained from star-elimination.

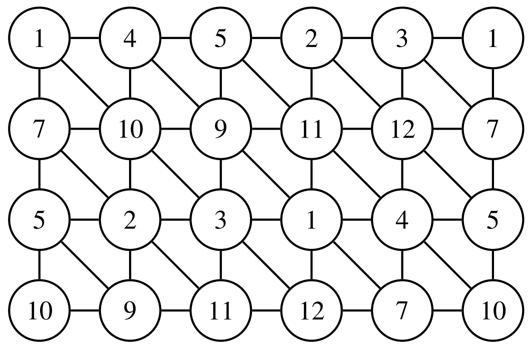

Figure 11. Subgraph of Ω_T with 12-TDL. Alternate rows of vertices use the same five labels.

We also find the $\chi_{\text{td}}(G)$ for the infinite complete binary tree graph, defined as the graph starting with a single vertex and where every vertex has two children, repeating infinitely.

Theorem 12. *Let B be the graph of the infinite complete binary tree. We have $\chi_{\text{td}}(B) = 7$.*

Proof. Since this graph is 3-regular except at the base of the tree, we can use the same method as with the infinite hexagonal lattice to obtain that $\chi_{\text{td}}(B) \geq 7$.

To show that $\chi_{\text{td}}(G) \leq 7$, we give a specific labeling for this graph. We will supply a pair of numbers for each label used in the labeling. We label the base of the tree with 1. We then label each pair of vertices in the tree use the following set of rules. Each pair after the arrow indicates the two corresponding labels for the next pair of vertices:

$$1 \to 3, 5, \quad 2 \to 5, 7, \quad 3 \to 2, 7, \quad 4 \to 6, 7, \quad 5 \to 4, 7, \quad 6 \to 1, 2, \quad 7 \to 1, 6.$$

It is straightforward to check that the resulting labeling of the tree is a TDL. □

One question that is implicitly raised by the results in this section is when an infinite graph has a finite total difference labeling number. Clearly if a graph has arbitrarily high-degree vertices then it cannot have a finite total difference labeling number (for that matter it cannot even have a finite coloring number). This is essentially the only circumstance where the total difference labeling fails to exist. In particular we have:

Theorem 13. *Let G be a countable, infinite graph, where $\Delta(G)$ is finite. Then $\chi_{\text{td}}(G)$ is defined. Moreover, let M be the largest element of a minimal well-spaced row with $\Delta(G)^2+1$ elements. Then $\chi_{\text{td}}(G) \leq M$.*

Proof. Assume G is a countable, infinite graph with $\Delta(G)$ finite. Let S be a well-spaced row with greatest element M and let the vertices of G be $v_1, v_2, v_3, \ldots$.

We set N_i to be the set of all vertices that are distance 1 or distance 2 away from v_i. Note that there are at most $\Delta(G)^2$ elements in any N_i.

Then we label the vertices inductively, assigning to each v_i the smallest element of S that has not yet been assigned to any label in N_i. Since N_i itself has at most $\Delta(G)^2$ elements, and S has $\Delta(G)^2+1$ elements, we can always find such a label. The labeling that results is a total difference labeling. Since the labeling uses a well-spaced row, we just need to check that there are no sandwiches and no duplicate adjacent vertices, but both are ruled out since no vertex v_j is ever labeled the same as any other vertex in N_j. □

The labeling given in Theorem 13 has some drawbacks. It is frequently much less efficient than the ideal labeling for an infinite graph. For example, this labeling scheme would tell you that $\chi_{\text{td}}(\Omega_S) \leq 53$, since the most efficient well-spaced row with $\Delta(\Omega_S)^2+1=4^2+1=17$ elements has largest element 53. Second, the specific labeling you get from this is not canonical but depends sensitively on the order the vertices are listed.

Note also that although we have only stated Theorem 13 for countable graphs, this phrasing is essentially a matter of convenience to make the proof straightforward and avoid any issues involving the axiom of choice. The theorem is also valid for larger cardinality graphs as long as one assumes choice.

4. Cloning and hypercubes

Let Q_d be the graph obtained from the d-dimensional hypercube. That is, Q_0 is a single vertex, Q_1 is the graph of two connected vertices, Q_2 is the square, and so on.

Straightforward computations establish $\chi_{\text{td}}(Q_0)=1$, $\chi_{\text{td}}(Q_1)=3$, $\chi_{\text{td}}(Q_2)=5$, $\chi_{\text{td}}(Q_3)=7$, and $\chi_{\text{td}}(Q_4)=9$. At this point, one might wish to guess that $\chi_{\text{td}}(Q_5)=11$. Alas, this is not the case. In fact, $\chi_{\text{td}}(Q_5)=10$. We cannot give a complete description of $\chi_{\text{td}}(Q_d)$ but will prove an estimate using a concept we call cloning.

Given a graph G with vertices $g_1, g_2, g_3, \ldots$ (with possibly infinitely many vertices), we will define the *clone of* G to be the Cartesian product of G with K_2. This is the graph made by $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ with edges given by the following:

(1) x_i is connected to x_j if and only if g_i is connected to g_j.

(2) y_i is connected to y_j if and only if g_i is connected to g_j.

(3) x_i is connected to y_j if and only if $i=j$.

In other words, to clone a graph G, make a copy of G and connect each vertex of G to its corresponding copy. Notice that the series of hypercubes is obtained by cloning the trivial graph. Given a graph G, we will write $\text{cl}(G)$ to be its clone.

Thus, for example, if G is the square graph Q_2, then $\mathrm{cl}(G)$ is the 3-dimensional cube graph Q_3, and $\mathrm{cl}(\mathrm{cl}(G))$ is the 4-dimensional hypercube graph Q_4, and so on.

Estimating $\chi_{\mathrm{td}}(G)$ for hypercubes will rely on the following lemma:

Lemma 14. *If G is a graph, then $\chi_{\mathrm{td}}(\mathrm{cl}(G)) \le 2\chi_{\mathrm{td}}(G)+1$.*

Proof. Let $f(G)$ be a TDL function of G. Let $H = \mathrm{cl}(G)$. We label H with labeling function h defined as follows: For each vertex g_i in G, let $h(x_i) = f(g_i)$ and $h(y_i) = f(g_i)+X_t d+1$ (where, recall, the x_i and y_j are corresponding copies of vertices). We claim that this is a TDL with largest label $2\chi_{\mathrm{td}}(G)+1$. It is immediately apparent that $h(H)$ has maximum label $\chi_{\mathrm{td}}(G)+\chi_{\mathrm{td}}(G)+1 = 2\chi_{\mathrm{td}}(G)+1$. Thus, we just need to check that there are no doubles, sandwiches or staircases.

There are no doubles among the set of x_i because they are directly labeled from our total difference labeling from G. There are no doubles among the y_i because the smallest value of any $h(y_i)$ is greater than half the largest value of the y_i labels. There is no double going from an x_i to a y_i because all the labels for the y_i are greater than twice the largest x_i-value.

There are no sandwiches among the x_i because they again have the same labeling as in G. There are no sandwiches among the y_i because each of the y_i are all the same values as the x_i but increased by a constant. There are three possible ways there could be a sandwich with a combination of the x_i and the y_i. First, there could be a sandwich of the form x_a, y_b, x_c, but this cannot happen because there is no set of connected vertices of that form. Second, there could be a sandwich of the form y_a, x_b, y_c, but again there are no connected vertices of that form. The third possible form of a sandwich is x_a, z, y_b, where z may be either an x- or a y-vertex. But such a sandwich would require that $h(x_a) = h(y_b)$, which is never true.

There are no staircases among the x_i because the x_i inherited their labels from the labeling of G. There are no staircases among the y_i because their labels are all a constant up from the x_i labels. There are no staircases involving both x_i and y_i because the difference between the largest x_i and smallest y_i is larger than the smallest difference between any x_i (which is also the smallest difference between any y_i). □

We can apply Lemma 14 inductively to the hypercubes to get the following:

Lemma 15. *For all d we have $\chi_{\mathrm{td}}(Q_d) \le 2^{d+1}-1$.*

It is pretty clear that Lemma 15 gives what is often a weak bound. For example, we know that χ_{td} of the square lattice is 8. This would tell us that clone of the square lattice has χ_{td} at most 17. But the clone of the square lattice is a subgraph of the cubic lattice, where we know χ_{td} is at most 13. In this case, Lemma 14 is giving a significant overestimate of the actual value of χ_{td}.

When is it that $\chi_{\mathrm{td}}(\mathrm{cl}(G)) = 2\chi_{\mathrm{td}}(G)+1$? Are there infinitely many graphs with this property? The only graph we are aware of where this bound is exactly equal is when G is a lone vertex. For certain families of graphs we can prove explicitly that this bound is weak. In the case of a path graph, the clone is just a cycle graph, and so that this bound is not best possible follows immediately from Propositions 4 and 5. The next two results show that this bound is not best possible for complete graphs and star graphs.

Proposition 16. *Let $n \geq 3$. Then $\chi_{\text{td}}(\text{cl}(K_n)) \leq 2\chi_{\text{td}}(K_n)$.*

Proof. Assume we have the graph K_n with its most efficient total difference labeling. This labeling for K_n is then the minimal well-spaced row with n elements. We now consider the graph $\text{cl}(K_n)$ with one copy of K_n labeled $v_1, v_2, \ldots, v_n$ and the other labeled $u_1, u_2, \ldots, u_n$, and with v_i connected to u_i for i satisfying $1 \leq i \leq n$. We assign to each of the v_i one of the labels from our well-spaced row from K_n, in increasing order, so $f(v_1)$ is smallest label and $f(v_n)$ has our largest label. We note that since this is a well-spaced row we must either be missing 1 as a label or must be missing 2 as a label in our well-spaced row.

Assume that 2 is missing in our well-spaced row. Then for $1 \leq i \leq n-1$ we set $f(u_i) = f(v_i) + \chi_{\text{td}}(K_n)$, and set $f(u_n) = 2$. We cannot have a double or a triple among the v_i because the v_i form a well-spaced row. Since $f(v_n) > 4$, we cannot have a double between v_n and u_n, and we cannot have a double or a staircase between the other u_i and v_i by the same logic as we had with our basic cloning argument in the proof of Lemma 14. Finally, we cannot have a staircase involving u_n since the only possible staircase involving 2 is $1-2-3$ which cannot occur here.

The case where 1 is the missing label is similar. □

One obvious question is if one has a random graph (in the Erdős–Rényi sense), is it true that this lemma is, with probability 1, very weak?

Conjecture 17. *For any $\epsilon > 0$, given the Erdős–Rényi random graph model, with probability 1, for a random graph G and its clone H, $\chi_{\text{td}}(H) \leq (1+\epsilon)\chi_{\text{td}}(G)$.*

Recall that the clone of a graph is its Cartesian product with K_2 and that the Cartesian product of two graphs G_1, G_2 is the graph $H = G_1 \square G_2$ satisfying:

(1) The set of vertices of H is the Cartesian product of $V(G_1)$ and $V(G_2)$.

(2) A vertex $(g_1, g_2) \in V(H)$ is adjacent to another vertex $(g_1', g_2') \in V(H)$ if and only if either $g_1 = g_1'$ and g_2 is adjacent to g_2' in G_2, or $g_2 = g_2'$ and g_1 is adjacent to g_1' in G_1.

Using similar logic as Lemma 14 one can prove:

Theorem 18. *Let $H = K_m \square G$ for some m. Then $\chi_{\text{td}}(H) \leq m\chi_{\text{td}}(G) + m(m-1)/2$.*

It seems natural to ask the following:

Question 19. *Is there a function $f(x, y)$ such that for any graphs G_1 and G_2 we have $\chi_{\text{td}}(G_1 \square G_2) \leq f(\chi_{\text{td}}(G_1), \chi_{\text{td}}(G_2))$?*

Another natural question is to ask whether it is always true for any graph G that $\chi_{\text{td}}(\text{cl}(G)) > \chi_{\text{td}}(G)$. This inequality is in fact false for the Petersen graph. If G is the Petersen graph, then $\chi_{\text{td}}(\text{cl}(G)) = \chi_{\text{td}}(G) = 10$. This raises the following question:

Question 20. *Is there a graph G such that, for any nonempty graph H, we have $\chi_{\text{td}}(H \square G) > \chi_{\text{td}}(G)$?*

Other notions of graph products exist, including the tensor product and strong product, and it may be natural to ask similar questions about how total difference labeling interacts with those products.

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Received: 2021-08-03 Revised: 2022-11-06 Accepted: 2022-11-13

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